The extended Killing form:

By demanding that

$$\langle [Z,X],Y\rangle + \langle X,[Z,Y]\rangle = 0$$

holds for elements $X, Y, Z \in \hat{g}$, one can derive:

- $\langle X^a[n], X^b[m]\rangle = \delta^{ab}\delta_{n+m,0}$
- $\langle X^a[n], \hat{k}\rangle = 0$ and $\langle \hat{k}, \hat{k}\rangle = 0$
- $\langle X^a[n], L_0\rangle = 0$ and $\langle L_0, \hat{k}\rangle = -1$

The only unspecified norm is $\langle L_0, L_0\rangle$ which one takes by definition to be

- $\langle L_0, L_0\rangle = 0$

Let the components of the vector $\hat{\lambda}$ be the eigenvalues of a state that is simultaneous eigenvector of the generators:

$$\hat{\lambda} = (\hat{\lambda}(H^1), \hat{\lambda}(H^2), \ldots, \hat{\lambda}(H^r); \hat{\lambda}(\hat{k}); \hat{\lambda}(-L_0))$$

$$\longrightarrow \hat{\lambda} = (\lambda; k_\lambda; n_\lambda)$$

Scalar product induced by extended Killing form

$$(\hat{\lambda}, \hat{\mu}) = (\lambda, \mu) + k_\lambda n_\mu + k_\mu n_\lambda$$

$\hat{\lambda}$ is called "affine weight"

Weights in adjoint representation are "roots":

$$\hat{\beta} = (\beta; 0; n) \quad (\hat{k} \text{ commutes with all generators})$$

$\rightarrow$ scalar product: $(\hat{\beta}, \hat{\alpha}) = (\beta, \alpha)$

Affine root associated with generator $E^{\alpha}[n]$:

$$\hat{\alpha} = (\alpha; 0; n) \quad n \in \mathbb{Z}, \quad \alpha \in \Delta \qquad (*)$$

Let

$$\delta = (0; 0; 1)$$

$\rightarrow$ $n\delta$ is root associated with $H^i[n]$

Notation : $\alpha := (\alpha; 0; 0)$

$\rightarrow$ $(*)$ can be expressed as

$$\hat{\alpha} = \alpha + n\delta$$

The full set of roots is

$$\hat{\Delta} = \{\alpha + n\delta \mid n \in \mathbb{Z}, \alpha \in \Delta\} \cup \{n\delta \mid n \in \mathbb{Z}, n \neq 0\}$$

root δ has length $(\delta, \delta) = 0$

$\rightarrow$ called "imaginary" root.

Simple roots, the Cartan matrix and Dynkin diagrams

Basis of simple roots is given by finite simple roots α_i and

$$\alpha_0 := (-\theta; 0; 1) = -\theta + \delta$$

where θ is the highest root of g.

$\longrightarrow$ set of positive roots is:

$$\hat{\Delta}_+ = \{\alpha + n\delta \mid n > 0, \alpha \in \Delta\} \cup \{\alpha \mid \alpha \in \Delta_+\}$$

Indeed, for $n > 0$ and $\alpha \in \Delta$,

$$\alpha + n\delta = \alpha + n\alpha_0 + n\theta = n\alpha_0 + (n-1)\theta + (\theta + \alpha)$$

$\longrightarrow$ coefficients of expansion in terms of finite simple roots non-negative

Given a set of affine simple roots, we can define "extended Cartan matrix"

$$\hat{A}_{ij} = (\alpha_i, \hat{\alpha}_j) \qquad 0 \leq i, j \leq r$$

where affine coroots are given by

$$\hat{\alpha}_i^\vee = \frac{2}{|\hat{\alpha}|^2}(\alpha_i, 0; n) = \frac{2}{|\alpha|^2}(\alpha; 0; n) = (\alpha^\vee; 0; \frac{2}{|\alpha|^2} n)$$

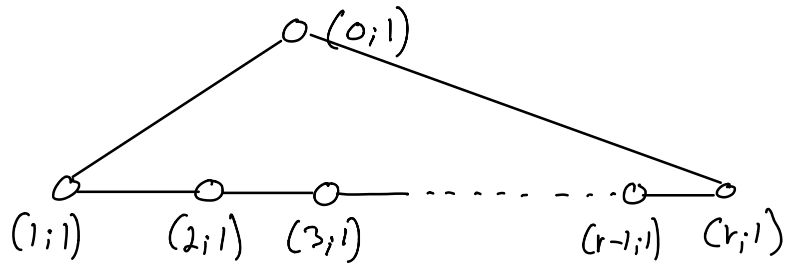
As for simple roots, the hat is omitted over the simple coroots, e.g.

$$\alpha_0^\vee = \alpha_0 \qquad \alpha_i^\vee = (\alpha_i^\vee;0;0) \qquad i \neq 0$$

$\hat{A}_{ij}$ contains extra row and column:

$$(\alpha_0, \alpha_j^\vee) = -(\theta, \alpha_j^\vee) = -\sum_{i=1}^{r} a_i(\alpha_i, \alpha_j^\vee)$$

→ gives rise to "extended Dynkin diagram":
Dynkin diagram of $\hat{g}$ is obtained from the one of g by addition of extra node representing α_0 → linked to α_i-nodes by $\hat{A}_{0i}\hat{A}_{i0}$ lines, e.g. for $\hat{A}$:



The zeroth mark a_0 is defined to be 1.

$$\rightarrow a_0^\vee = a_0 \frac{|\alpha_0|^2}{2} = 1$$

By construction the extended Cartan matrix satisfies:

$$\sum_{i=0}^{r} a_i \hat{A}_{ij} = \sum_{i=0}^{r} \hat{A}_{ij} a_j^\vee = 0$$

The imaginary root can be written as:

$$\delta = \sum_{i=0}^{r} a_i \alpha_i = \sum_{i=0}^{r} a_i^\vee \alpha_i^\vee$$

Similarly, the dual Coxeter number reads

$$h^\vee = \sum_{i=0}^{r} a_i^\vee$$

Fundamental weights:

Fundamental weights $\{\hat{\omega}_i\}$, $0 \leq i \leq r$ are dual to simple coroots

$$\longrightarrow \quad \hat{\omega}_i = (\omega_i \,;\, a_i^\vee \,;\, 0) \qquad (i \neq 0)$$

$\hat{K}$ eigenvalue is fixed by the condition

$$(\hat{\omega}_i , \alpha_0^\vee) = 0 \qquad (i \neq 0)$$

The zeroth fundamental weight must have zero scalar product with all α_i's and satisfy $\quad (\hat{\omega}_0 , \alpha_0^\vee) = 1$

$$\longrightarrow \quad \hat{\omega}_0 = (0 ; 1 ; 0)$$ "basic fundamental weight"

Thus we get

$$\hat{\omega}_i = a_i^\vee \hat{\omega}_0 + \omega_i \qquad \text{where } \omega_i = (\omega_i ; 0 ; 0)$$

Affine weights can be expanded as

$$\hat{\lambda} = \sum_{i=0}^{r} \lambda_i \hat{\omega}_i + l\delta \quad , \quad l \in \mathbb{R}$$

$$\longrightarrow \quad k := \hat{\lambda}(\hat{k}) = \sum_{i=0}^{r} a_i^\vee \lambda_i \qquad \text{"level"}$$

This relation can also be derived by :

$$(\hat{\lambda}, \delta) = k_{\hat{\lambda}} n_\delta + \cancel{n_{\hat{\lambda}} k_\delta}^{\,0} = \hat{\lambda}(\hat{k})$$

$$= \sum_{i=0}^{r} a_i^\vee (\hat{\lambda}, \alpha_i^\vee) = \sum_{i=0}^{r} a_i^\vee \lambda_i$$

where we used $\delta = \sum_{i=0}^{r} a_i^\vee \alpha_i^\vee$.

$$\longrightarrow \quad \lambda_0 = \hat{\lambda}(\hat{k}) - \sum_{i=1}^{r} a_i^\vee \lambda_i \qquad (a_0 = 1)$$

Thus we get

$$\lambda_0 = k - (\lambda, \theta)$$

Affine weights will generally be given by

$$\hat{\lambda} = [\lambda_0, \lambda_1, \ldots, \lambda_r]$$

(this notation does not keep track of L_0 eigenvalue)

For instance,

$$\hat{\omega}_0 = [1, 0, \ldots, 0], \quad \hat{\omega}_1 = [0, 1, \ldots, 0], \quad \hat{\omega}_r = [0, 0, \ldots, 1]$$

Moreover, we have

$$\alpha_i = [\hat{A}_{i0}, \hat{A}_{i1}, \ldots, \hat{A}_{ir}]$$

Finally, the "affine Weyl vector" is defined as

$$\hat{\rho} = \sum_{i=0}^{r} \hat{\omega}_i = [1,1,\ldots,1], \quad \hat{\rho}(\hat{k}) = h^\vee.$$

<u>Integrable highest-weight representations:</u>

IHWRs of $\hat{g}$ are those representations whose projections onto the su(2)-algebra associated with root $\hat{\alpha}$ are finite

su(2) - algebra:

$$E = E^{-\alpha}[n], \quad F = E^{\alpha}[-n], \quad H = \hat{k} - \alpha \cdot H[0]$$

$$\Rightarrow \quad [E,F] = H, \quad [H,E] = 2E, \quad [H,F] = -2F$$

One usually takes $n=1$.

Then, finiteness of su(2) representations gives that any weight $\hat{\lambda}'$ in $M_{\hat{\lambda}}$ satisfies:

$$(\hat{\lambda}', \alpha_i^\vee) = -(p_i - q_i) \qquad i = 0,1,\ldots,r \qquad (**)$$

for some positive integers p_i, q_i.

$\longrightarrow \lambda_i' \in \mathbb{Z}, \qquad i = 0, 1, \dots, r$

For the highest weight $\hat{\lambda}$, all p_i's are zero, and therefore

$$\lambda_i \in \mathbb{Z}_+, \qquad i = 0, 1, \dots, r$$

$$\Longrightarrow \quad \lambda_0 = k - (\lambda, \theta) \in \mathbb{Z}_+$$

$\longrightarrow$ k is a positive integer:

$$k \in \mathbb{Z}_+, \qquad k \geq (\lambda, \theta)$$

An affine weight for which all Dynkin labels are non-negative integers is said to be "dominant" $\longrightarrow$ set denoted by P_k^+.

For the highest weight state, $v_{\hat{\lambda}}$, conditions $(**)$ are equivalent to:

$$\left(E^{\theta}[-1]\right)^{k - (\lambda, \theta) + 1} v_{\hat{\lambda}} = 0$$

$$(\hat{\lambda}, \alpha_0^\vee) = k - (\lambda, \theta) = -(p_0 - q_0) = q_0$$

$\longrightarrow$ $F^{q_0} v_{\hat{\lambda}}$ gives lowest weight of $su(2)$-rep.

$\longrightarrow$ proof of Prop. 5

$\square$